

Newton's second law is true when the momentum is calculated relative to an inertial reference frame.

An inertial frame is a reference frame such that any particles or bodies moving in that frame that have no net force or net moment applied, will not accelerate to the observer.

$$\bar{F} = \frac{d\bar{p}}{dt} \Big|_{IF} \quad \bar{p} = m(U\hat{i} + V\hat{j} + W\hat{k})$$

both  $U, V, W$  and  $\hat{i}, \hat{j}, \hat{k}$   
change in time when viewed  
from IF.

$$\frac{d\bar{p}}{dt} \Big|_{BF} = m(\dot{U}\hat{i} + \dot{V}\hat{j} + \dot{W}\hat{k})$$

body  
fixed

What is the relation between  $\frac{d\bar{p}}{dt} \Big|_{BF}$  and  $\frac{d\bar{p}}{dt} \Big|_{IF}$ ?

$$\frac{d\bar{p}}{dt} \Big|_{IF} = \frac{d\bar{p}}{dt} \Big|_{BF} + \bar{\omega} \times \bar{p}$$

↑  
angular velocity  
of BF in IF

so what is  $\bar{\omega} \times \bar{p}$ ?

$$\bar{\omega} = p\hat{i} + q\hat{j} + r\hat{k} \quad \bar{p} = m(U\hat{i} + V\hat{j} + W\hat{k})$$

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$$\det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & q & r \\ mu & mv & mw \end{vmatrix} = m \begin{bmatrix} qw - rv \\ -(pw - ru) \\ pv - qu \end{bmatrix} = \bar{\omega} \times \bar{p}$$

$$\bar{F} = \frac{d\bar{p}}{dt} \Big|_{BF} = \frac{d\bar{p}}{dt} \Big|_{BF} + \bar{\omega} \times \bar{p}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = M \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix}$$

Newton's Second Law  
for a rigid body  
translation expressed  
via the body  
fixed coordinate  
system

## Euler's Second Law (for rotational dynamics)

$$\bar{\tau} = \frac{d\bar{H}_c}{dt}$$

↑                          ← angular momentum  
net moment              of the body  
acting on              wrt the  
body                      mass center

$$\bar{H}_c = I_c \bar{\omega}$$

↑  
moment of inertia  
matrix wrt mass  
center and body  
fixed axes

=

Moment of Inertia

$$I_c = \begin{bmatrix} 3 \times 3 \\ I_{xx} \ I_{xy} \ I_{xz} \\ I_{yx} \ I_{yy} \ I_{yz} \\ I_{zx} \ I_{zy} \ I_{zz} \end{bmatrix}$$

it describes the mass  
distribution of the  
rigid body



$$I_{xx} = Mr^2$$

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \int (y^2 + z^2) dm$$

$$I_{yy} = \int (x^2 + z^2) dm$$

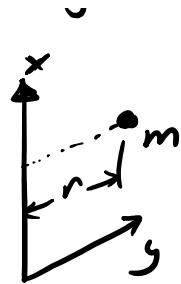
$$I_{zz} = \int (x^2 + y^2) dm$$

$$I_{xy} = - \int xy dm = I_{yx}$$

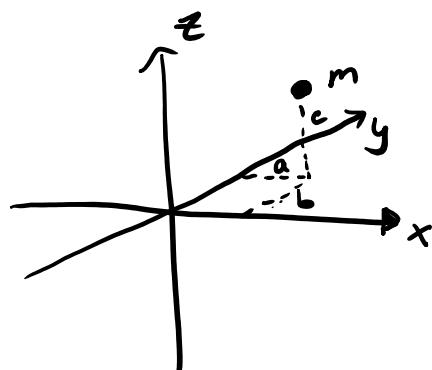
$$I_{yz} = - \int yz dm = I_{zy}$$

$$I_{zx} = - \int zx dm = I_{xz}$$

moments  
of  
inertia



$I_{xx} = Mr^2$   
moment of inertia  
about  $x$



$$\begin{aligned} I_{xx} &= m(b^2 + c^2) \\ I_{yy} &= m(a^2 + c^2) \\ I_{zz} &= m(a^2 + b^2) \end{aligned}$$

if  $c=0$  then we have symmetry  
about the  $x-y$  plane

$$I_{yz} = 0 \text{ and } I_{zx} = 0$$

$$I_{xx} = mb^2 \quad I_{yy} = ma^2$$

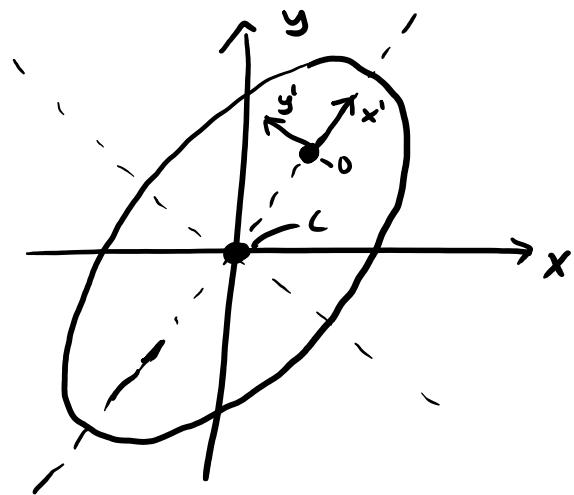
$$\begin{aligned} I_{xy} &= -mab \\ I_{yz} &= -mbc \\ I_{zx} &= -mca \end{aligned}$$

$I_c$  has 6 unique values in general, 3 MoI  
+ 3 PoI

If  $x, y, z$  are the so called principal axes then  
all PoI are zero:  $I_{xy} = I_{xz} = I_{yz} = 0$ .

The principal axes are the axes such that  
the mass is distributed symmetrically relative to  
those axes. Every object has these axes  
and they will align with geometric symmetry

and they will align with geometric symmetry  
if the mass is uniformly distributed.



$$I_c^{x'y'z'} \Rightarrow \text{no PoI}$$

$$I_o^{x'y'z'} \Rightarrow \text{has PoI}$$

=

Back come to:

$$\bar{\tau} = \left. \frac{d \bar{H}_c}{dt} \right|_{IF}$$

$$\bar{H}_c = \begin{bmatrix} I_{xx} & - & - \\ - & I_{yy} & - \\ - & - & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\left. \frac{d \bar{H}_c}{dt} \right|_{BF} = \begin{bmatrix} I_{xx}\dot{p} + I_{xy}\dot{q} + I_{xz}\dot{r} \\ I_{yx}\dot{p} + I_{yy}\dot{q} + I_{yz}\dot{r} \\ I_{zx}\dot{p} + I_{zy}\dot{q} + I_{zz}\dot{r} \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

$$\bar{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\bar{\tau} = \left. \frac{d \bar{H}_c}{dt} \right|_{IF} = \left. \frac{d \bar{H}_c}{dt} \right|_{BF} + \underbrace{\bar{\omega} \times \bar{H}_c}_{?}$$

$$\bar{\omega} \times \bar{H}_c = \begin{bmatrix} q(I_{zx}p + I_{zy}q + I_{zz}r) - r(I_{yx}p + I_{yy}q + I_{yz}r) \\ -p(I_{zx}p + I_{zy}q + I_{zz}r) + r(I_{xx}p + I_{xy}q + I_{xz}r) \end{bmatrix}$$

$$\begin{aligned} & -p(I_{xx}\dot{r} + I_{yy}\dot{q} + I_{zz}\dot{r}) + q(I_{xx}\dot{r} + I_{yy}\dot{q} + I_{zz}\dot{r}) \\ & p(I_{yx}\dot{p} + I_{yy}\dot{q} + I_{yz}\dot{r}) - q(I_{xy}\dot{p} + I_{yy}\dot{q} + I_{xz}\dot{r}) \end{aligned}$$

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} + I_{xy}\dot{q} + I_{xz}\dot{r} + q(I_{zx}\dot{p} + I_{zy}\dot{q} + I_{zz}\dot{r}) - r(I_{yx}\dot{p} + I_{yy}\dot{q} + I_{yz}\dot{r}) \\ I_{xx}\dot{p} + I_{xy}\dot{q} + I_{xz}\dot{r} + q(I_{zx}\dot{p} + I_{zy}\dot{q} + I_{zz}\dot{r}) - r(I_{yx}\dot{p} + I_{yy}\dot{q} + I_{yz}\dot{r}) \\ I_{yy}\dot{q} + I_{yz}\dot{r} + p(I_{xy}\dot{p} + I_{yy}\dot{q} + I_{xz}\dot{r}) - r(I_{yx}\dot{p} + I_{yy}\dot{q} + I_{yz}\dot{r}) \end{bmatrix}$$

If principle axes, equations simplify:

$$L = I_{xx}\dot{p} + (I_{zz} - I_{yy})qr$$

$$M = I_{yy}\dot{q} + (I_{xx} - I_{zz})pr$$

$$N = I_{zz}\dot{r} + (I_{yy} - I_{xx})pq$$