

Deriving Equations of Motion with Lagrange's Method

Lagrange developed an energy based method that generates equations of motion that are equivalent to those from the Newton-Euler approach.

Advantages:

- only magnitudes of velocities
- constraint forces that do no work can be ignored
- systematically produces equations that are always in the same form

Disadvantages

- system must be described by a minimal set of coordinates
- forces & torques must be mapped to the minimal set of coordinates

Lagrange's Equation For a Single Rotating Rigid Body with single coordinate θ :

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{\theta}} - \frac{\partial T^*}{\partial \theta} + \frac{\partial V}{\partial \theta} = \underline{\quad}$$

T^* : total kinetic energy of the rigid body as a function of θ and $\dot{\theta}$

V : total potential energy of the system (conservative forces that do work: springs & gravity)

$\underline{\underline{Q}}_0$: generalized forces acting on the rigid body
 (non-conservative forces that do work: e.g. friction)

Kinetic energy of single rigid body in planar motion

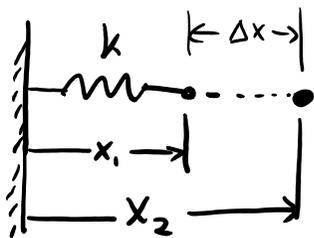
$$T^* = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \dot{\theta}^2$$

translational rotational

total mass of R.B. magnitude of velocity of the mass center moment of inertia about the mass center magnitude of the angular velocity of the R.B.

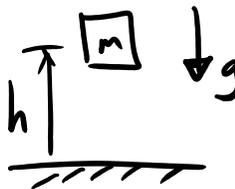
Typical potential energy terms:

Linear spring

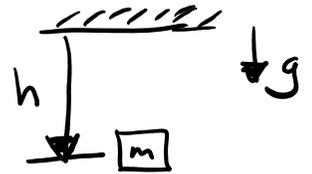


$$V = \frac{1}{2} k (\Delta x)^2 = \frac{1}{2} k (x_2 - x_1)^2$$

Gravity



$$V = mgh$$

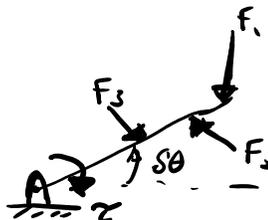


$$V = -mgh$$

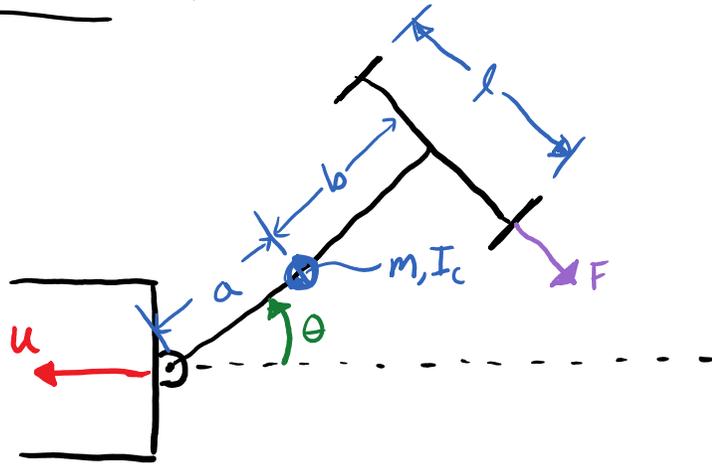
Generalized Forces

Imagine an infinitesimally small change $\delta\theta$ in the coordinate θ and the small amount of work δW done by any non-conservative forces & torques that resist the change in θ :

$$\delta W = \underline{\underline{Q}}_0 \delta\theta$$



Trailer (with realistic forces)



Assumptions

- vehicle is @ constant speed U
- frictionless hitch
- only planar
- single rigid body
- tires are rolling (no brakes, no driving torque..)

Time varying variables:

θ : trailer angle

F : combined lateral force of both tires

- tire forces $F = C_{\alpha} \alpha$

Constant parameters

U : vehicle speed

m : trailer mass

I_c : yaw moment of inertia

a : distance from hitch to mass center

b : distance from mass center to axle

l : track width

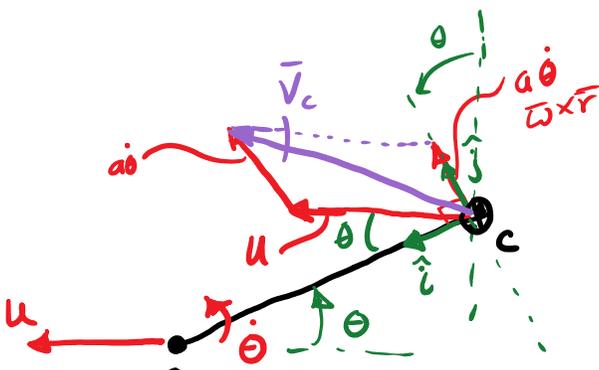
Basic motion, equilibrium state

$$\theta = 0, \dot{\theta} = 0, F = 0$$

T^* kinetic energy

$$T^* = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \dot{\theta}^2$$

θ : generalized coordinate



$$\bar{v}_c = \bar{v}_0 + \bar{\omega} \times \bar{r}^{0 \rightarrow c}$$

$$\bar{v}_c = U \cos \theta \hat{i} +$$



$$\vec{v}_c = U \cos \theta \hat{i} + (U \sin \theta + a \dot{\theta}) \hat{j}$$

$$v_c^2 = |\vec{v}_c|^2 = (U \cos \theta)^2 + (U \sin \theta + a \dot{\theta})^2$$

$$v_c^2 = U^2 + a^2 \dot{\theta}^2 + 2Ua \dot{\theta} \sin \theta$$

Note: Law of Cosines

