

$$T^* = \frac{1}{2}m(U^2 + a^2\dot{\theta}^2 + 2U_a\dot{\theta}\sin\theta) + \frac{1}{2}I_c\dot{\theta}^2$$

$$V = 0$$

$$\frac{\partial T^*}{\partial \dot{\theta}} = ma^2\dot{\theta} + mUa\sin\theta + I_c\dot{\theta}$$

$$② \frac{\partial T^*}{\partial \theta} = \cancel{mUa\dot{\theta}\cos\theta}$$

$$\frac{\partial V}{\partial \theta} = 0$$

$$① \frac{d}{dt} \left( \frac{\partial T^*}{\partial \dot{\theta}} \right) = ma^2\ddot{\theta} + \cancel{mUa\cos\theta\dot{\theta}} + I_c\ddot{\theta}$$

$\underbrace{\qquad\qquad\qquad}_{(ma^2 + I_c)\ddot{\theta}} \underbrace{-F(a+b)}_{F} \underbrace{(a+b)}_{\theta + \frac{(a+b)}{U}\dot{\theta}}$

$$(ma^2 + I_c)\ddot{\theta} = -F(a+b) = -C_2(\theta + \frac{(a+b)}{U}\dot{\theta})(a+b)$$

$$(ma^2 + I_c)\ddot{\theta} + C_2 \frac{(a+b)^2}{U}\dot{\theta} + C_2(a+b)\theta = 0$$

$\underbrace{(ma^2 + I_c)}_{\text{inertial term}} \underbrace{\frac{(a+b)^2}{U}}_{\text{damping coeff.}} \underbrace{C_2(a+b)}_{\text{stiffness coefficient}} \theta = 0$

$$J\ddot{\theta} + B\dot{\theta} + K\theta = 0$$

$$M\ddot{x} + Cx + Kx = 0$$



If  $J, B, K$  are all greater than zero the system is stable.

$$ma^2 + I_c > 0 \quad \checkmark$$

$$C_2 \frac{(a+b)}{U} > 0 \quad \checkmark$$

System is stable!

$$C_d(a+b) > 0 \quad \checkmark$$

as  $U \rightarrow \infty$   $\frac{C_d(a+b)}{U} \rightarrow 0$  thus at high speed we have

$$J\ddot{\theta} + K\theta = 0$$



$J\ddot{\theta} + B\dot{\theta} + K\theta = 0$  can also be represented as

$$\ddot{\theta} + 2\zeta\dot{\theta} + \omega_n^2\theta = 0$$

$$\omega_n = \sqrt{\frac{K}{J}} \quad \zeta = \frac{B}{2J\omega_n}$$

↑                      ↑  
natural frequency    damping ratio

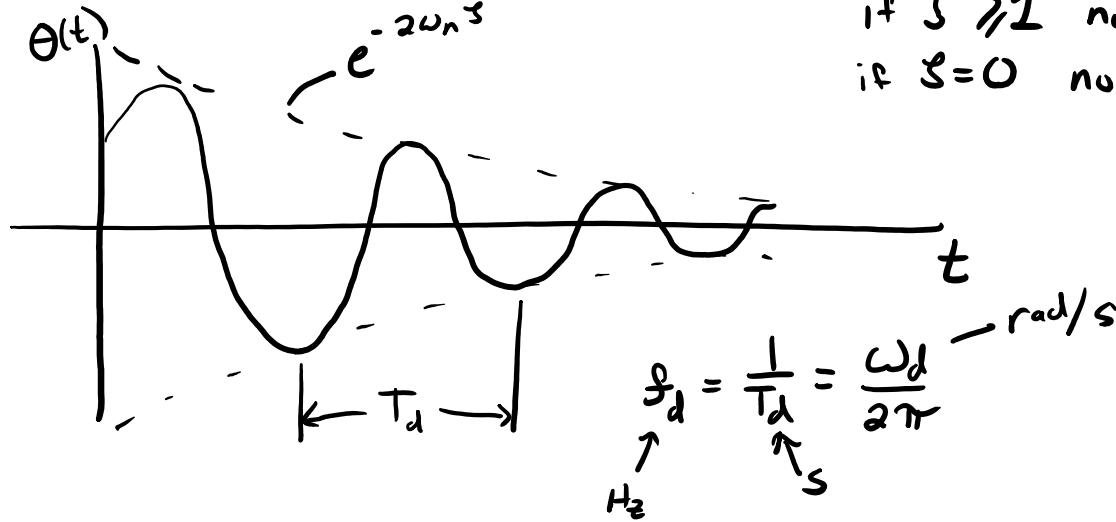
$$\theta(t) = A e^{-2\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

↑  
damped natural frequency

General solution for  $0 \leq \zeta < 1$

if  $\zeta \geq 1$  no oscillation  
if  $\zeta = 0$  no decay



$\dots, 2, 1, 0$

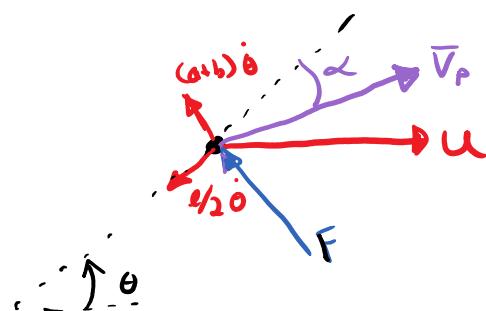
$$H_2 \rightarrow$$

$\omega_n^2 > 0$  for stability  
 $2\omega_n \zeta > 0$

$$\omega_n^2 = \frac{K}{J} = \frac{C_a(a+b)}{I_c + ma^2} > 0 \quad \checkmark$$

$$2\omega_n \zeta = 2\omega_n \frac{B}{2J\omega_n} = \frac{B}{J} = \frac{C_a(a+b)^2/U}{I_c + ma^2} > \checkmark$$

What if  $U < 0$ ?



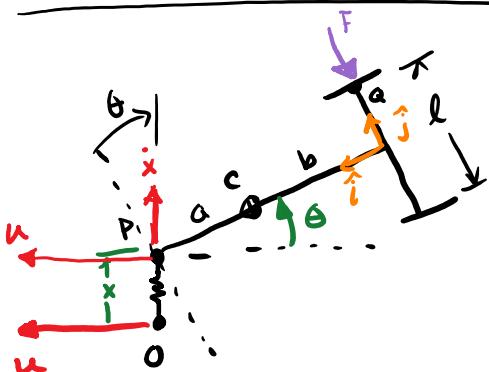
$$\alpha_p = \frac{-U\theta + (a+b)\dot{\theta}}{-U + \cancel{\frac{1}{2}\dot{\theta}}} = \theta - (a+b) \frac{\dot{\theta}}{U}$$

$$\bar{\ddot{\theta}} = F(a+b)$$

$$(ma^2 + I_c)\ddot{\theta} + \frac{C_a(a+b)^2}{U}\dot{\theta} - C_a(a+b)\theta = 0$$

$\underbrace{\quad}_{\text{always negative}}$

### Two Degree of Freedom Trailer Model



$$\bar{v}_c = \bar{v}_p + \bar{\omega} \times \bar{r}^{P \rightarrow C}$$

$$\bar{v}_c = (U\omega s\theta - \dot{x} \sin\theta) \hat{i} + (U\sin\theta + \dot{x} \cos\theta + a\dot{\theta}) \hat{j}$$

$$v_c^2 = (U\omega s\theta - \dot{x} \sin\theta)^2 + (U\sin\theta + \dot{x} \cos\theta + a\dot{\theta})^2$$

$$v^2 = u^2 + v^2 + a^2\dot{\theta}^2 + 2aU\dot{\theta}\sin\theta + 2a\dot{x}\dot{\theta}\cos\theta$$

$$r_0 = \sqrt{u^2 + \dot{x}^2 + \alpha^2 \dot{\theta}^2}$$

$$V_c^2 = u^2 + \dot{x}^2 + \alpha^2 \dot{\theta}^2 + 2u\dot{u}\dot{\theta}\sin\theta + 2\alpha\dot{x}\dot{\theta}\cos\theta$$

$$\bar{V}_q = (u\cos\theta - \dot{x}\sin\theta + \ell_b\dot{\theta})\hat{i} + (u\sin\theta + \dot{x}\cos\theta + (\alpha+b)\dot{\theta})\hat{j}$$