Two DoF Trailer Model Linear Equations of Motion

$$\bar{y} = \begin{bmatrix} \theta \\ x \end{bmatrix} \qquad M\ddot{\bar{y}} + C\dot{\bar{y}} + K\bar{y} = 0$$

$$M = \begin{bmatrix} I_c + ma^2 & ma \\ ma & m \end{bmatrix} \quad C = \begin{bmatrix} c_\alpha \frac{(a+b)^2}{u} & c_\alpha \frac{a+b}{u} \\ c_\alpha \frac{a+b}{u} & c_\alpha \frac{1}{u} \end{bmatrix} \quad K = \begin{bmatrix} c_\alpha (a+b) & 0 \\ c_\alpha & k \end{bmatrix}$$

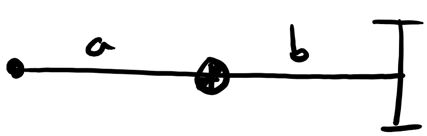
Characteristic equation:

$$\det \left| Ms^2 + Cs + K \right| = 0$$

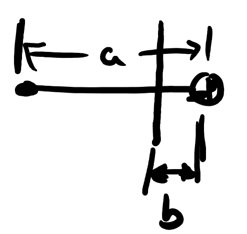
$$m I_c s^4 + (I_c + mb^2)(c_\alpha/u)s^3 + [(I_c + ma^2)k + mb\,c_\alpha]s^2 + [c_\alpha(a+b)^2 k/u]s + c_\alpha(a+b)k = 0$$

Routh-Hurwitz $\Rightarrow$ look for negative coefficients to s^n

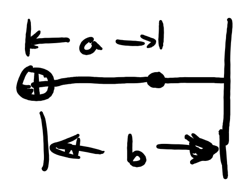
$I_c, m, c_\alpha, u, k > 0$



$b < 0$, $a > 0$



$a < 0$, $b > 0$



s^4 $\quad mI_c$ $\quad$ positive

s^3 $\quad \frac{(I_c + mb^2)c_\alpha}{u}$ $\quad$ positive

s^2 $\quad (I_c + ma^2)k + mb\,c_\alpha$ $\quad$ possibly negative

s^1 $\quad c_\alpha(a+b)^2 k/u$ $\quad$ positive

$s^1 \quad C_\alpha (a+b)^2 k/u$ positive

$s^0 \quad C_\alpha (a+b) k$ possibly be negative

$$(I_c + ma^2)k + mbC_\alpha < 0$$

$$b < \frac{-(I_c+ma^2)k}{mC_\alpha}$$ unstable!

$$C_\alpha (a+b)k < 0$$

$$b < -a$$ unstable!

If all coeffs are positive, then go to step 2 in R-H criterion.

Let's simplify the model by examining the behavior as $C_\alpha \to \infty$. As $C_\alpha \to \infty$ the model approaches the no-slip condition. This is the case for lower speeds.

$$\cancel{mI_c s^4} + (I_c + mb^2)(C_\alpha/u)s^3 + [\cancel{(I_c+ma^2)}k + mbC_\alpha]s^2 +$$

$$[C_\alpha(a+b)^2 k/u]s + C_\alpha(a+b)k = 0$$

$$\frac{I_c+mb^2}{u}s^3 + mbs^2 + (a+b)^2\frac{k}{u}s + (a+b)k = 0$$

3rd polynomial in s

$s^3 \quad I_c + mb^2$ positive

a_0 $\quad \dfrac{I_c + mb^2}{u}$ positive

a_1 $\quad mb$ possibly negative

a_2 $\quad (a+b)^2 \dfrac{k}{u}$ positive

a_3 $\quad (a+b)k$ possibly negative

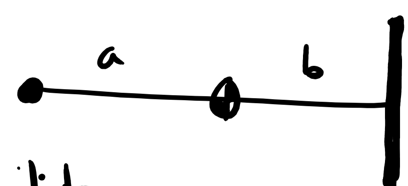
If $a, b > 0$ all coeff are positive, go to step 2 of RH:

For $\quad a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$

$a_1 a_2 > a_0 a_3$ the system is stable

$$mb(a+b)^{\cancel{2}} \frac{\cancel{k}}{\cancel{u}} > \frac{(I_c + mb^2)\cancel{(a+b)}\cancel{k}}{\cancel{u}}$$

$$\underline{\underline{mab > I_c}}$$ for stability



But the wrong placement of the mass center can cause instability at any speed!!

Lastly, increase the complexity of the 3^{rd} order model by adding rotary damping at the hitch.

$$\Xi_\theta = -F(a+b) - \underbrace{c\,\dot{\theta}}_{\text{linear rotary}}$$

linear rotary
viscous damping

$$C = \begin{bmatrix} c_\alpha \dfrac{(a+b)^2}{u} + c & c_\alpha \dfrac{a+b}{u} \\ c_\alpha \dfrac{a+b}{u} & c_\alpha \dfrac{1}{u} \end{bmatrix}$$

Follow the same procedure to arrive @ a 3rd poly.

$$\frac{I_c + mb^2}{u} s^3 + \left(\frac{c}{u} + mb\right) s^2 + \left(\frac{(a+b)^2 k}{u}\right) s + (a+b)k = 0$$

If all parameters > 0 then apply R-H table:

$$\frac{c(a+b)}{u} + mab > I_c \quad \text{for stability}$$

$$u < \underbrace{\frac{c\,(a+b)}{I_c - mab}}_{} \quad \text{for stability}$$

denominator is positive, in prior model
unstable at
all speeds

$$U_{crit} = \frac{c\,(a+b)}{I_c - mab}$$

boundary between stable $U < U_{crit}$
and unstable $U > U_{crit}$

Conclusions

- Trailer is always unstable when backing up.

- Trailer is stable at forward speeds below U_{crit}.
- Trailer can be unstable at speeds above U_{crit} if $mab < I_c$.