

For a multi-input, multi-output system (MIMO):

$M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} = B\bar{u}$ \Rightarrow apply the Laplace Transform to lead to a set of transfer functions that relate each input-output pair

For example, if

Inputs (\bar{u}): s_f, s_r Outputs (\bar{x}): V, r

we get 4 transfer functions.

Using Laplace Transform:

$$V(t) \Rightarrow V(s), \quad \dot{V}(t) \Rightarrow sV(s)$$

$$r(t) \Rightarrow r(s), \quad \dot{r}(t) \Rightarrow sr(s)$$

$$s_f(t) \Rightarrow s_f(s), \quad s_r(t) \Rightarrow s_r(s)$$

$$M \begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} + C \begin{bmatrix} v \\ r \end{bmatrix} = B \begin{bmatrix} s_f \\ s_r \end{bmatrix}$$

$$A = sM + C$$

$$\begin{bmatrix} ms + (c_f + c_r)/u & m \cdot m u + \frac{(a c_f - b c_r)}{u} \\ - & - \\ - & - \\ (a c_r - b c_f)/u & I_2 s + \frac{(a^2 c_f + b^2 c_r)}{u} \end{bmatrix} \begin{bmatrix} V(s) \\ r(s) \end{bmatrix} = \begin{bmatrix} c_f & c_r \\ - & - \\ a c_f & - b c_r \\ \hline m & R \end{bmatrix} \begin{bmatrix} s_f(s) \\ s_r(s) \end{bmatrix}$$

$$L \underbrace{A_v}_{\bar{A}_v} : \underbrace{u}_{\bar{A}_r} \rightarrow \underbrace{\text{current}}_{\bar{B}_{S_f}} \underbrace{\text{current}}_{\bar{B}_{S_r}}$$

$\det |A| = 0 \Rightarrow$ characteristic equation

$\Delta \triangleq \det |A| \Rightarrow$ denominator

Use Cramer's rule to calculate numerators of TFs:
 for output $V(s)$ and input $\delta_f(s)$, replace
 \bar{A}_v in A with \bar{B}_{S_f} in B and take the
 determinant of the new matrix.

$$\frac{V(s)}{\delta_f(s)} = \frac{\det |\bar{B}_{S_f} : \bar{A}_r|}{\Delta} = \frac{I_z C_f s - m a C_f U + (a+b) b C_r C_f / U}{\Delta \rightarrow 2^{\text{nd}} \text{ order}}$$

$$\frac{V(s)}{\delta_r(s)} = \frac{\det |\bar{B}_{S_r} : \bar{A}_r|}{\Delta} = \frac{I_z C_r s + m b C_r U + (a+b) a C_r C_f / U}{\Delta}$$

$$\frac{r(s)}{\delta_f(s)} = \frac{\det |\bar{A}_v : \bar{B}_{S_f}|}{\Delta} = \frac{m a C_f s + (a+b) C_f C_r / U}{\Delta}$$

$$\frac{r(s)}{\delta_r(s)} = \frac{\det |\bar{A}_v : \bar{B}_{S_r}|}{\Delta} = \frac{-m b C_r s - (a+b) C_f C_r / U}{\Delta}$$

- By setting $s = \omega i$ where $i = \sqrt{-1}$ we can then find the frequency dependent magnitude and phase ratios from the transfer function. This characterizes the frequency response (if $\delta_f(t) = \sin(\omega t)$)

ratios from the transfer function
the frequency response (if $\delta_f(t) = \sin(\omega t)$)

- The numerators are first order in s and the denominator is second order in s . This means the output amplitude is attenuated at high frequency inputs (i.e. low pass filter).
- The steady state input-output relationship can be found by setting $s=0$.

$$\left. \frac{r(s)}{\delta_f(s)} \right|_{s=0} = \frac{(a+b)C_f C_r / U}{(a+b)^2 C_f C_r / U^2 + m(bC_r - aC_f)} = \frac{U}{(a+b) + \frac{m(bC_r - aC_f)U^2}{(a+b)C_f C_r}}$$

Define $K \triangleq \frac{m(bC_r - aC_f)}{(a+b)C_f C_r}$ understeer coefficient

$K=0$ neutral steer

$K > 0$ understeer

$K < 0$ oversteer

$$\left. \frac{r(s)}{\delta_f(s)} \right|_{s=0} = \frac{U}{(a+b) + KU^2}$$

When $s=0$ (or $\omega=0$), TF value is also called the "zero frequency gain"; multiplicative ratio of the input and output.

$$G_z \triangleq \left. \frac{r(s)}{\delta_f(s)} \right|_{s=0}$$

yaw rate - front steer gain

$$G_{rf} \triangleq \left. \frac{r(s)}{\delta_f(s)} \right|_{s=0} \quad \text{yaw rate - front steer gain}$$

$$G_{rr} \triangleq \left. \frac{r(s)}{\delta_r(s)} \right|_{s=0} = \frac{-U}{(a+b) + KU^2}$$

