

$$(I_i + mh^2)\ddot{\theta} - mgh\dot{\theta} = -mh(\dot{V} + rU)$$

↓

equation for
inverted pendulum

effective roll torque from
steering

$$(I_i + mh^2)\ddot{\theta} - mgh\dot{\theta} = - \frac{mh}{a+b} (b\dot{\delta}_f U + S_f U^2 + a\dot{\delta}_r U - S_r U^2)$$

If $U=0$ we no effective torque, bicycle will fall.

At high speeds the steer angles affect torque more. At low speeds the steer rates affect torque more.

Take the Laplace Transform.

$$[(I_i + mh^2)s^2 - mgh]\Theta(s) = - \frac{mh}{a+b} [(sbU + U^2)\delta_f(s) + (saU - U^2)\delta_r(s)]$$

Set $\delta_r(s) = 0$

$$\frac{\Theta(s)}{\delta_f(s)} = \frac{-\frac{mhU}{a+b}(sb+U)}{(I_i + mh^2)s^2 - mgh}$$

\leftarrow 1st order $\underline{\text{filter}}$
 \leftarrow 2nd order

Poles: $s = \pm \sqrt{\frac{mgh}{I_i + mh^2}}$ 2 real roots, 1 pos, 1 neg

Zeros: $s = -\frac{U}{b}$ 1 negative real zero

If $S_f(s) = 0$

$$\underline{\Theta(s)} = -\frac{mhU}{a+b} (sa - U)$$

$$\frac{\Theta(s)}{\delta_r(s)} = \frac{-\frac{mhU}{a+b}(sa-U)}{(I_i+mh^2)s^2 - mgh}$$

zeros: $s = \frac{U}{a}$ 1 positive real root
 non-minimum phase behavior
 poles: same as above

$$\frac{\Theta(s)}{\delta_f(s)} = \frac{-\frac{mhU}{a+b}(sb+U)}{(I_i+mh^2)s^2 - mgh} \cdot \frac{\frac{1}{mgh}}{\frac{1}{mgh}}$$

$$\frac{\Theta(s)}{\delta_f(s)} = -\frac{\underbrace{\frac{U^2}{g(a+b)}}_K \left(\frac{b}{U}s + 1\right)}{\underbrace{\frac{I_i+mh^2}{mgh}s^2 - 1}_{\tau_1^2}} = -\frac{K(\tau_2 s + 1)}{\tau_1^2 s^2 - 1}$$

Let's see if proportional negative feedback controller will stabilize the system.

Θ_d : desired roll angle

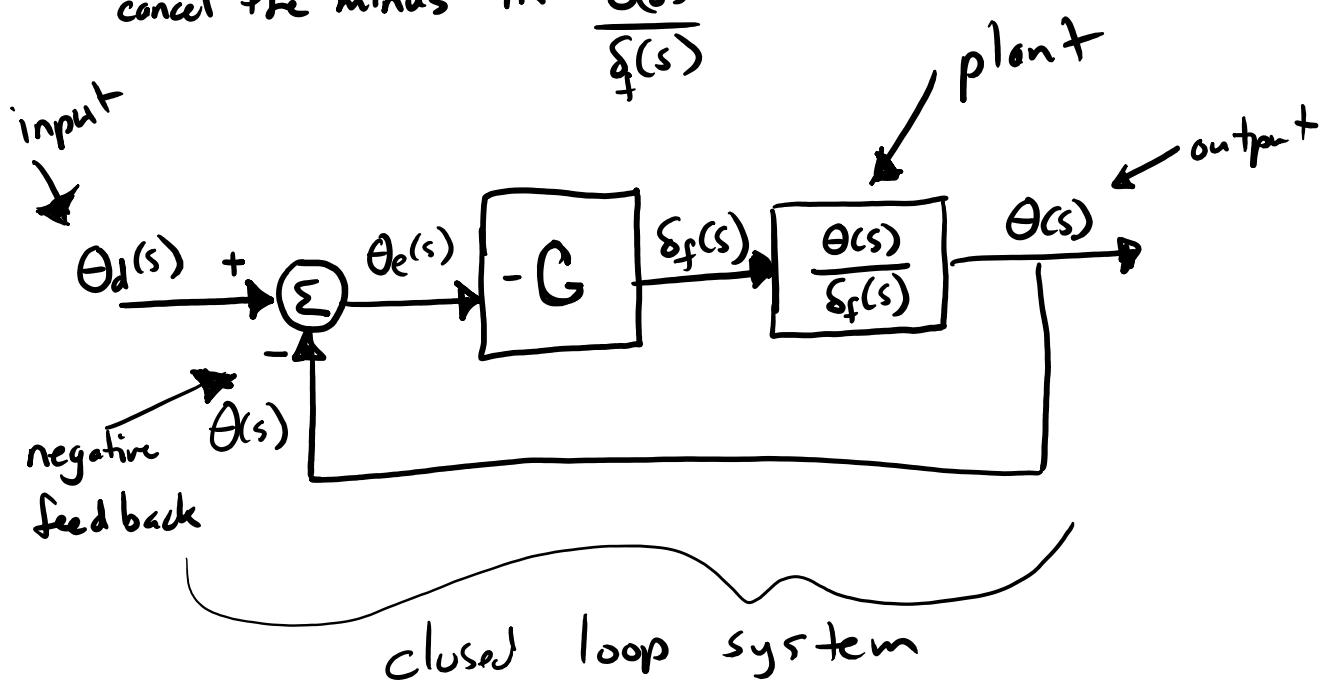
Θ_e : error of the roll angle wrt Θ_d

$$\Theta_e = \Theta_d - \Theta$$

$$\delta_f = -G(\Theta_d - \Theta) \xrightarrow{\text{negative feedback}}$$

$$\delta_f = -G(\theta_d - \theta)$$

cancel the minus in $\frac{\theta(s)}{\delta_f(s)}$



$$\frac{\theta(s)}{\theta_d(s)} \Rightarrow$$

$$\frac{\theta(s)}{\delta_f(s)} = \frac{\theta(s)}{-G(\theta_d(s) - \theta(s))} = \frac{-K(\zeta_2 s + 1)}{\zeta_1^2 s^2 - 1}$$

$$\downarrow$$

$$\theta[\zeta_1^2 s^2 - 1] = GK(\zeta_2 s + 1)[\theta_d - \theta]$$

$$\frac{\theta(s)}{\theta_d(s)} = \frac{GK(\zeta_2 s + 1)}{\zeta_1^2 s^2 + GK\zeta_2 s + GK - 1}$$

closed loop transfer function

char. eq. of
closed loop system

$$\gamma_1^2 s^2 + GK\gamma_2 s + GK - 1 = 0$$

$$\gamma_1^2 = \frac{I_i + mh^2}{mgh} \quad \text{positive always}$$

$$GK\gamma_2 = \frac{GU^2}{g(a+b)} \frac{b}{U} \quad \begin{aligned} \text{positive } G \text{ gives pos. coeff} \\ G > 0 \end{aligned}$$

$$GK - 1 = \frac{GU^2}{g(a+b)} - 1$$

$$G > \frac{g(a+b)}{U^2}$$

As $U \rightarrow 0, G \rightarrow \infty$

Countersteering

Create a transfer function for ^{the} closed loop system with $S_f(s)$ as an output and $\theta_d(s)$ as the input.

$$\frac{S_f(s)}{\theta_d(s)} = \frac{S_f(s)}{\cancel{\theta(s)}} \cdot \frac{\cancel{\theta(s)}}{\theta_d(s)} = \frac{-G(\gamma_1^2 s^2 - 1)}{\gamma_1^2 s^2 + GK\gamma_2 s + (GK-1)}$$

\uparrow \uparrow
 plant closed
 loop

Assume $GK > 1 \Rightarrow$ system is stable.