

- Write your top 3 review topics for exam in order of importance
- HW #9 Due Friday (all problems)
- Two missing HWs will be excused
- Lab 5 due Monday
- Review on Friday

$$s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21} = 0$$

2nd order polynomial in s : the characteristic equation

The order of the polynomial corresponds to the size of A .

An n th order polynomial from an $n \times n$ A matrix.

An n th " " has n roots (solutions for s)

Use the quadratic formula:

$$s_{1,2} = \frac{a_{11} + a_{22}}{2} \pm \frac{\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

Depending on the values of $a_{11}, a_{22}, a_{21}, a_{12}$ there several possible solutions:

$$s_{1,2} = \lambda \pm \omega i \sqrt{-1}$$

a complex conjugate pair with a real part, λ , and an imaginary part ωi

$$s_1 = \lambda_1, s_2 = \lambda_2$$

two unique real values

$$s_{1,2} = \pm \omega i$$

complex conjugate pair with no real part

$$s_{1,2} = 0$$

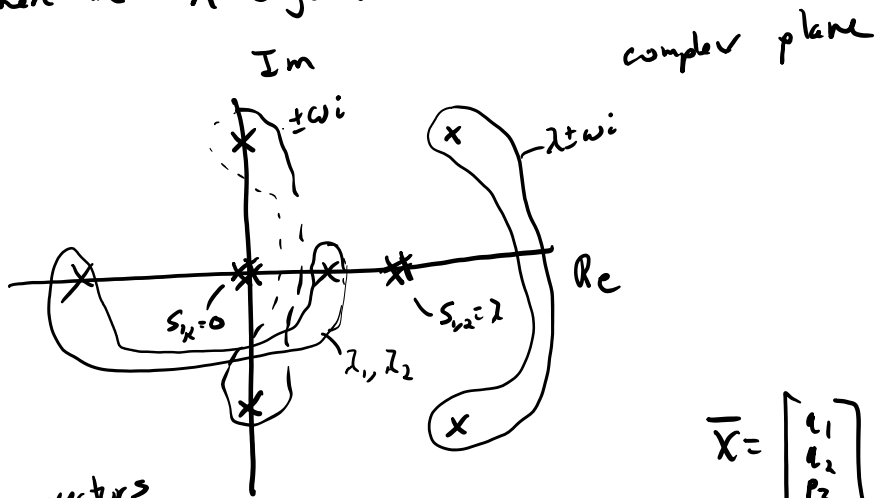
zeros

~

non-unique real values

$$s_{1,2} = \lambda$$

$s_{1,2}$ are called the eigenvalues of A . For an $n \times n$ A there are n eigenvalues.



$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \end{bmatrix} e^{st}$$

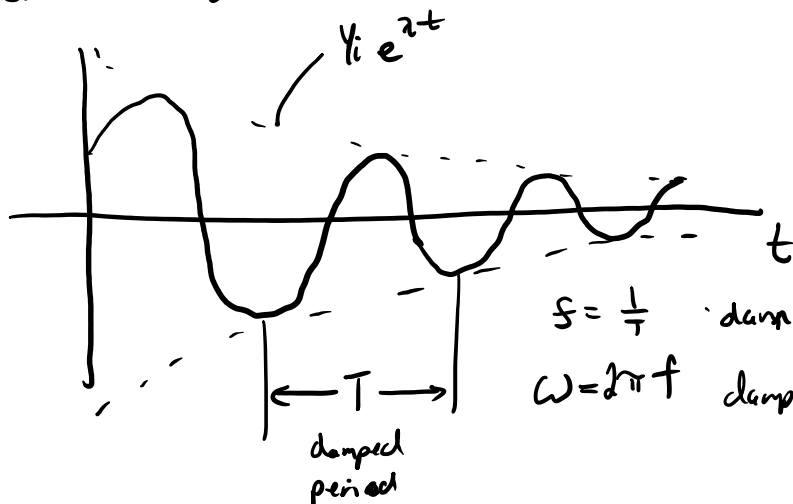
$$\bar{X} = \bar{X} e^{st} = \bar{X} e^{(\lambda \pm \omega i)t}$$

(use Euler's Formula)

$$\bar{X} = \bar{Y} e^{\alpha t} \sin(\omega t + \phi)$$

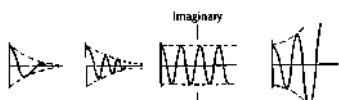
\bar{Y} is amplitude vector } determined by initial conditions
 ϕ phase angle

For a single state $x_i(t) = Y_i e^{\alpha t} \sin(\omega t + \phi)$



$$\zeta = \frac{\alpha}{\omega} \text{ damped frequency Hz}$$

$$\omega = 2\pi f \text{ damped frequency rad/s}$$



If any $\lambda > 0$ unstable system
 If all $\lambda < 0$ stable system

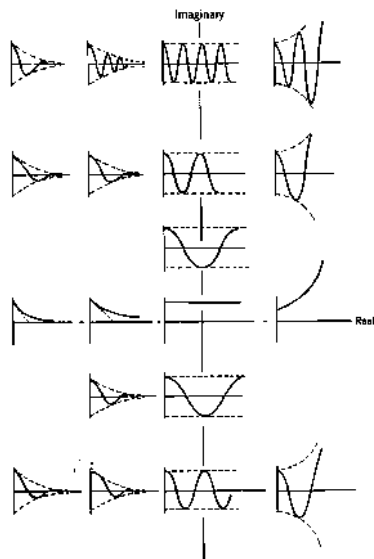
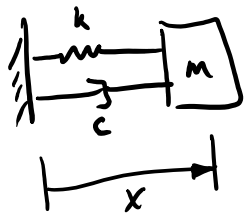


FIGURE 6.5. Correlation of time response with complex eigenvalues.

If all $\lambda < 0$ stable system
 If any $\lambda = 0$ marginally stable system
 and all others are < 0
 System is unstable if the eigenvalues are in the "right half plane"



second order linear ODE

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\left. \begin{aligned} v &= \dot{x} \\ m\dot{v} + cv + kx &= 0 \end{aligned} \right\} \text{2 first order linear ODEs}$$

$$\dot{v} = -\frac{c}{m}v - \frac{k}{m}x$$

$$\dot{\mathbf{x}} = \mathbf{v}$$

$$\begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} -c/m & -k/m \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} v \\ x \end{bmatrix}$$

$$\det |s\mathbf{I} - \mathbf{A}| = 0$$

$$\det \begin{vmatrix} s + c/m & -k/m \\ 1 & s \end{vmatrix} = 0$$

$$(s + c/m)s + k/m = 0$$

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

introduce

$$\omega_n = \sqrt{\frac{k}{m}}$$

↑

$$\zeta = \frac{c}{2m\omega_n}$$

↑

$$\Rightarrow s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

natural frequency

damping ratio

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}i$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{1-\zeta^2}i$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_d i$$

$$x(t) = Y_x e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

↳ damped natural frequency

$$0 \leq \zeta < 1$$

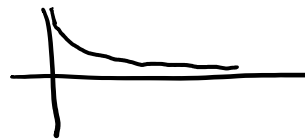
underdamped



$$s_{1,2} = -\zeta\omega_n \pm \omega_d i$$

$$\zeta = 1$$

critically damped



$$s_{1,2} = -\zeta\omega_n$$

$$\zeta > 1$$

overdamped



$$s_{1,2} = -\zeta\omega_n$$