- Lab 4 due tomorrow
- HW #9 due Friday March 13
- Lab 5 due Monday March 16
- Review Thursday March 12 lecture
- Final Wed. March 18 3:30 PM

# Chapter 6: Analysis of Linear System

State space form of a set of linear ordinary differential equations.

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$
$$\bar{y} = C\bar{x} + D\bar{u}$$

A: state matrix

B: input matrix

C: output matrix

D: feed forward matrix

$\bar{x}$: state vector

$\bar{u}$: input vector

$\bar{y}$: output vector

We've been numerically integrating $\dot{\bar{x}}(t)$ to finding $\bar{x}(t)$.

$$\bar{x} = \int_{t_0}^{t_f} \dot{\bar{x}}\,dt = \int_{t_0}^{t_f} (A\bar{x} + B\bar{u})\,dt$$

ode45 does this part!

If the system is linear we can also find $\bar{x}$ using analytical methods. The first step is to find solution $\dot{\bar{x}} = A\bar{x}$, the homogeneous solution.

Set $\bar{u} = 0$ for the unforced system.

$$\dot{\bar{x}} = A\bar{x}$$

$$\dot{\bar{x}} = A\bar{x}$$

Guess solution

$$\bar{x} = \bar{\bar{X}} e^{st} \qquad \dot{\bar{x}} = \bar{X} s e^{st}$$

$$\bar{\bar{X}} s e^{st} = A\bar{\bar{X}} e^{st} \quad \text{algebraic equation}$$

$$\rightarrow \bar{\bar{X}} e^{st} [sI - A] = 0 \qquad I : \text{identity matrix}$$

$e^{st} = 0$ can't be zero for all time

$\bar{X} = 0$ trivial solution

$[sI - A] = 0$ not a solution for arbitrary A

$$\bar{\bar{X}} [sI - A] = 0 \quad \text{non-trivial solution!}$$

If $\underbrace{\det|sI - A| = 0}_{\text{characteristic equation}}$ then we solutions for s

The characteristic equation is a polynomial in s and the roots of the polynomial are the solutions for s.

$b_1 s^n + b_2 s^{n-1} + \ldots + b_n s^{n-n} = 0$ if I have n states

the polynomial will be of order $\underline{n}$.

Example of 2x2 A matrix $\Rightarrow$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\det \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = 0$$

$$\det \begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix} = 0$$

$$(s - a_{11})(s - a_{22}) - a_{12} a_{21} = 0$$

$$\boxed{s^2 - (a_{11} + a_{22}) s + a_{11} a_{22} - a_{12} a_{21} = 0}$$

2nd order polynomial in $s$. "characteristic equation"

We have a quadratic polynomial so:

$$s_{1,2} = \frac{a_{11} + a_{22}}{2} \pm \frac{\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}$$

Depending on the values of $a_{11}, a_{22}, a_{12}, a_{21}$ there are a variety of solutions:

$s_{1,2} = \lambda \pm \omega i$ — $i = \sqrt{-1}$ a complex conjugate pair with real part and imaginary part

$s_1 = \lambda_1$, $s_2 = \lambda_2$ two unique real values

$s_{1,2} = \pm \lambda$ a pair of equal and opposite real values

$s_{1,2} = 0$ zeros

$s_{1,2} = \lambda$ repeated real value

$s_{1,2} = \pm \omega i$ complex conjugate pair with only

$s_{1,2} = \pm \omega i$ complex conjugate pair with only imaginary part

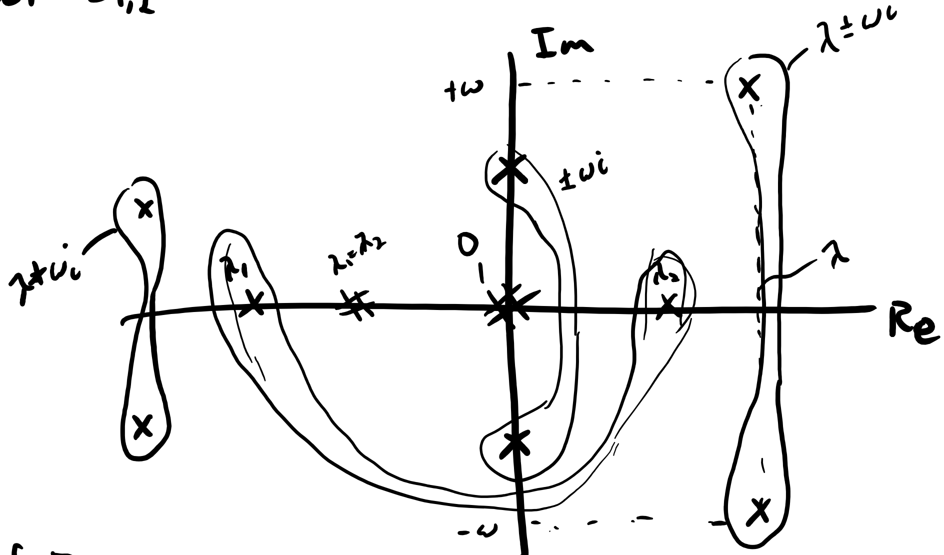
$s_{1,2}$ are called the <u>eigenvalues of $A$</u>. For $n \times n$ $A$ matrix there will be $n$ eigenvalues of the general form $s = \lambda \pm \omega i$,

$$\bar{x} = \bar{X} e^{(\lambda \pm \omega i)t}$$

Using Euler's formula, $e^{ix} = \cos x + i \sin x$, the above can be written as:

$$\boxed{\bar{x}(t) = \bar{Y} e^{\lambda t} \sin(\omega t + \phi)}$$

plot $s_{1,2}$ on the imaginary plane:



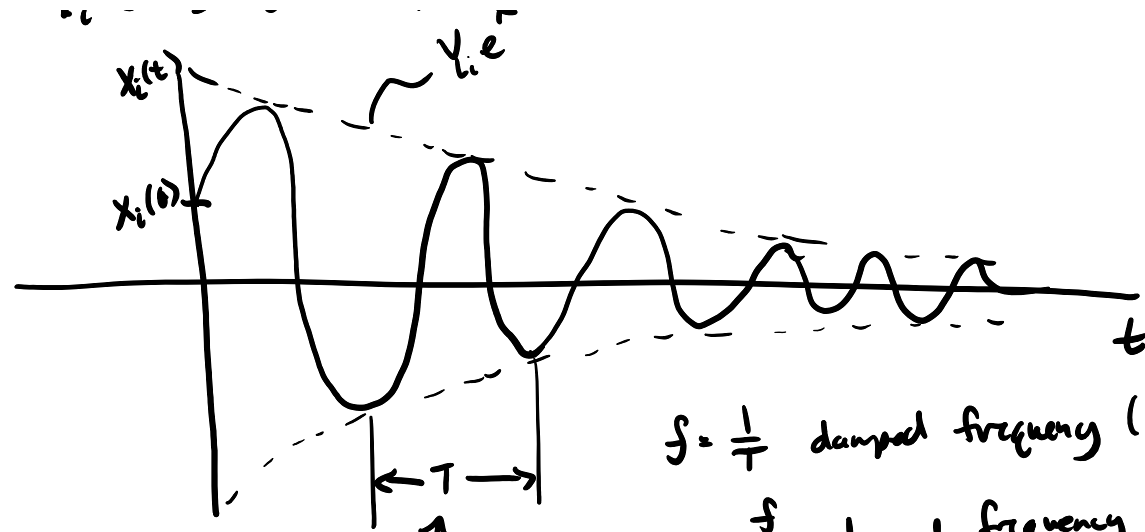

$\lambda > 0$ growth

$\lambda < 0$ decay

$\omega = 0$ no oscillation

$$\bar{x}(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_i = Y_i e^{\lambda t} \sin(\omega t + \phi)$$




$f = \frac{1}{T}$ damped frequency (Hz)

$\omega = \frac{f}{2\pi}$ damped frequency (rad/s)

$\tau = \frac{1}{\lambda}$ time constant (s)

time to decay to ~63% of the original value

$\boxed{\text{Look @ Figure 6.8!!}}$

If <u>any</u> $\lambda > 0 \Rightarrow$ unstable system

If all $\lambda < 0 \Rightarrow$ stable system

If all $\lambda < 0$ and one or more $\lambda = 0 \Rightarrow$ marginally stable system

Thursday, March 5, 2020 11:06 AM



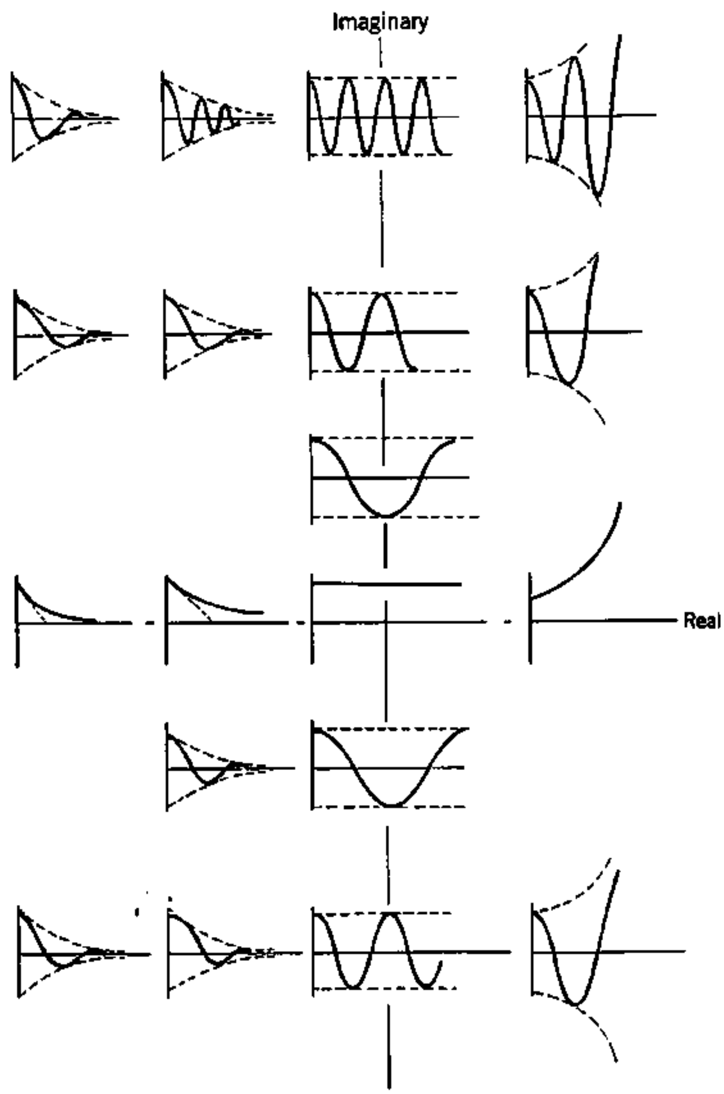



FIGURE 6.8. Correlation of time response with complex eigenvalues.