Motion Constraints

Even if $\bar{q}$ is minimal, the generalized speeds $u_1, \dots, u_n$ may not be independent. (This property of the system not the choice of the u's). If the u's are not independent the system S is subject to motion constraints (non-holonomic constraints).

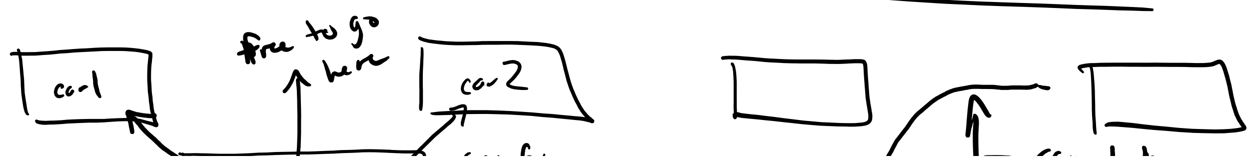
There is a special kind of non-holonomic systems called a <u>simple non-holonomic system</u> where:

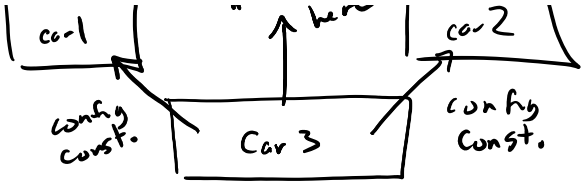
$$\underset{\text{dependent G.S.}}{u_r} = \sum_{s=1}^{p} \underbrace{A_{rs}}_{\text{linear dependence}} \underset{\text{independent G.S.}}{u_s} + B_r \qquad \underbrace{r = p+1, \dots, n}_{m \text{ items}}$$

$$\underset{\#\text{ ind. G.S.s}}{p} = \underset{\substack{\#\text{ min. G.C.s} \\ \& \\ \text{total } \# \text{ of G.S.}}}{n} - \underset{m \text{ dep G.S.s}}{m}$$

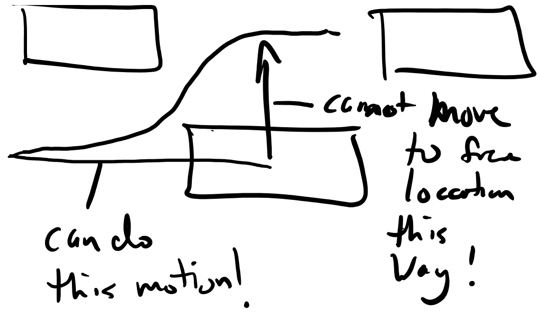
$$\bar{u} = \left.\begin{bmatrix} u_1 \\ \vdots \\ u_p \\ u_{p+1} \\ \vdots \\ u_n \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} p \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} m \end{matrix} \right\} n$$

- Configuration constraints
  - reduces # of independent G.C's
  - Holomic constraints
  - constrain where the system can go
- essential Motion constraints
  - reduces # DoF and # independent G.S's
  - Non-holomic constraints
  - constrain how the system can go when you want it to





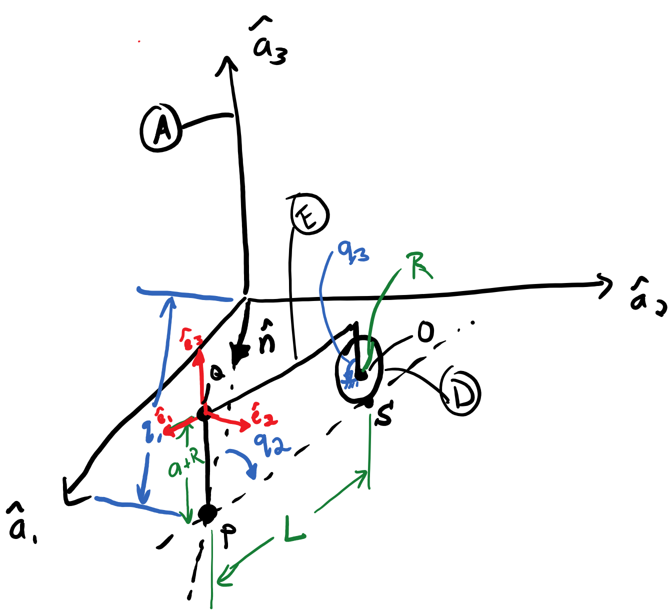
can't go where cars 1,2 are!



motion constraints

Motion Constraint Example

Simple trailer roll w/o slip on a ground plane



q_1, q_2, q_3 are min. GCs.

$n = 3$

$\hat{e}_3 = \hat{a}_3$

P is moved only in $\hat{n}$ direction.

$$\hat{n} = c_2 \hat{e}_1 + s_2 \hat{e}_2$$

$${}^A\bar{V}^P = \dot{q}_1 \hat{n} \qquad {}^A\bar{V}^Q = \dot{q}_1 \hat{n}$$

$${}^A\bar{V}^O = {}^A\bar{V}^Q + {}^A\bar{\omega}^E \times \bar{r}^{O/Q}$$

$$= \dot{q}_1 \hat{n} - \dot{q}_2 \hat{e}_3 \times (-L\hat{e}_1 - a\hat{e}_3)$$

$${}^A\bar{V}^O = \dot{q}_1 c_2 \hat{e}_1 + (\dot{q}_1 s_2 + L\dot{q}_3)\hat{e}_2$$

${}^A\bar{V}^S = 0$ if there is pure rolling

S is fixed in D

$${}^A\bar{V}^S = {}^A\bar{V}^O + {}^A\bar{\omega}^D \times \bar{r}^{S/O}$$

$${}^A\bar{V}^S = {}^A\bar{V}^O + (\dot{q}_3\hat{e}_2 - \dot{q}_2\hat{e}_3) \times (-R\hat{e}_3)$$

$${}^A\bar{V}^S = (\dot{q}_1 c_2 - R\dot{q}_3)\hat{e}_1 + (\dot{q}_1 s_2 + L\dot{q}_2)\hat{e}_2 = 0$$

$${}^A\bar{V}^S \cdot \hat{e}_1 = 0 = \dot{q}_1 c_2 - R\dot{q}_3 \quad (1)$$

$${}^A\bar{V}^S \cdot \hat{e}_2 = 0 = \dot{q}_1 s_2 + L\dot{q}_2 \quad (2)$$

2 motion constraints

Are these essential motion constraints?

$$\frac{df}{dt} = \frac{\partial f}{\partial q_1}\dot{q}_1 + \frac{\partial f}{\partial q_2}\dot{q}_2 + \frac{\partial f}{\partial q_3}\dot{q}_3 + \frac{\partial f}{\partial t} = 0$$

(1) $\frac{\partial f}{\partial q_1} = \cos q_2 \qquad \frac{\partial f}{\partial q_2} = 0 \qquad \frac{\partial f}{\partial q_3} = -R$

$\frac{\partial f}{\partial q_2 \partial q_1} = -\sin q_2 \qquad \frac{\partial f}{\partial q_1 \partial q_2} = 0 \qquad \times$ doesn't commute essential!!

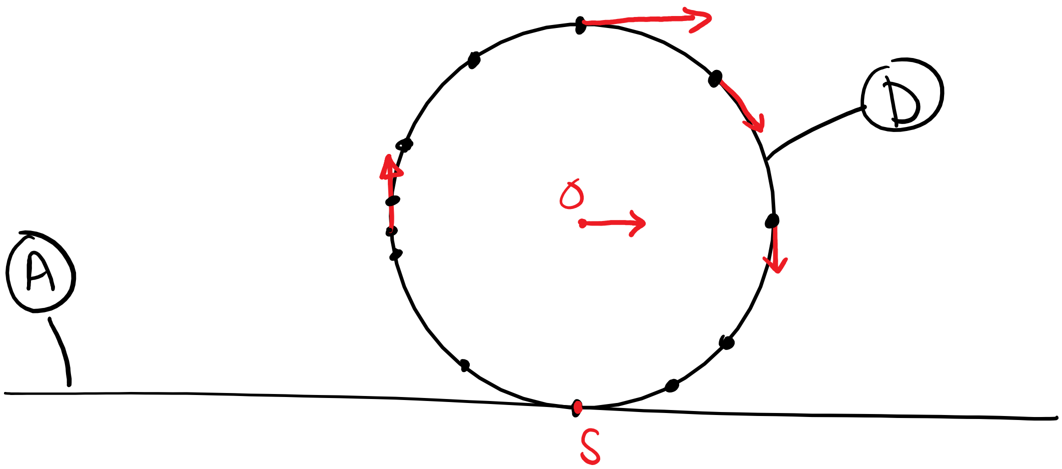
$\frac{\partial f}{\partial q_1 \partial q_3} = 0 \qquad \frac{\partial f}{\partial q_3 \partial q_1} = 0 \qquad \checkmark$

$\frac{\partial f}{\partial q_2 \partial q_3} = 0 \qquad \frac{\partial f}{\partial q_3 \partial q_2} = 0 \qquad \checkmark$

(2) $\frac{\partial f}{\partial q_1} = \sin q_2 \qquad \frac{\partial f}{\partial q_2} = L$

$\frac{\partial f}{\partial q_1 \partial q_2} = 0 \qquad \frac{\partial f}{\partial q_2 \partial q_1} = \cos q_2 \qquad \times$ essential!

$g(u_1, \dots, u_n, q_1, \dots, q_n, t) = 0$ potential motion constraint



Summary of GC's, GS's, and constraints

- $n + n^*$ generalized coordinates used to describe a system S's configuration

- there may be dependendent generalized coordinates that arise from holonomic configuration constraints $\bar{f}(\underbrace{q_1, \dots, q_n}_{\text{ind.}}, \underbrace{q_{n+1}, \dots, q_{n+n^*}}_{\text{dep}}; t) = 0$, such that there are n indepedent GC's and n^* dependent GC's.

  $\bar{f} \in \mathbb{R}^{n^*}$ $\bar{f} : n^* \times 1$

- given the minimal # of GC's $\bar{q} = [q_1, \dots, q_n]^T$ you can define generalized speeds $\bar{u} = [u_1, \dots, u_n]^T$ such that

$$\bar{u} = Y \dot{\bar{q}} + \bar{Z} \quad \text{or} \quad u_r = \sum_{s=1}^{n} Y_{rs} \dot{q}_s + Z_r \quad r = 1, \dots, n$$

that

$$\bar{u} = Y\dot{\bar{q}} + \bar{Z} \qquad \text{or} \qquad u_r = \sum_{s=1}^{n} Y_{rs}\dot{q}_s + Z_r \quad r=1,\dots,n$$

matrix form   scalar form

Y must be invertible (Y^{-1}) for your choice of GS's to be legitimate, i.e. $\underbrace{\dot{\bar{q}} = Y^{-1}(\bar{u} - \bar{Z})}_{\text{Kinematical diff. eqn.}}$ for all time

- If no motion constraints, then $n = p$, i.e. # min. GC's = # DoF resulting in n independent u's, n independent q's, and thus $2n$ first order differential equations in time.

$$n\times 1 \quad \dot{\bar{u}} = ? \qquad \Rightarrow \text{come from } \bar{F} = m\bar{a}$$

$$n\times 1 \quad \dot{\bar{q}} = Y^{-1}(\bar{u} - \bar{Z})$$

- If there are m $\overbrace{\text{essential}}^{\text{not integrable}}$ motion constraints, the $p = n - m$ u's are independent and m u's are dependendent. There will then be only p 1st order ODE's for the u's:

$$p\times 1 \quad \dot{\bar{u}}_s = [\dot{u}_1, \dots, \dot{u}_p] = ?$$

$$n\times 1 \quad \dot{\bar{q}} = Y^{-1}(\bar{u} - \bar{Z})$$

The dependent speeds are found from:

$$\bar{u}_r = A_{rs}\bar{u}_s + \bar{B}_r \qquad \text{or} \qquad u_r = \sum_{s=1}^{p} A_{rs} u_s + B_r \quad r = p+1,\dots,n$$

matrix   scalar form

Partial Velocities and Partial Angular Velocities

$q_1, \dots, q_n$ GC's $\qquad\qquad$ $u_1, \dots, u_n$ GS's

In general we have lots of rigid bodies, points, and particles whose motion we are interested in.
All velocities and angular velocities in the system can be expressed uniquely as linear functions of the u_i's.

unconstrained/holonomic

$$\overline{\omega} = \sum_{r=1}^{n} \underbrace{\overline{\omega}_r u_r}_{\text{linear in the } u\text{'s}} + \overline{\omega}_t \qquad\qquad \overline{v} = \sum_{r}^{n} \underbrace{\overline{v}_r u_r}_{\text{linear in } u\text{'s}} + \overline{v}_t$$

remainder $\qquad$ remainder

rth partial anglular velocity $\qquad$ rth partial velocity

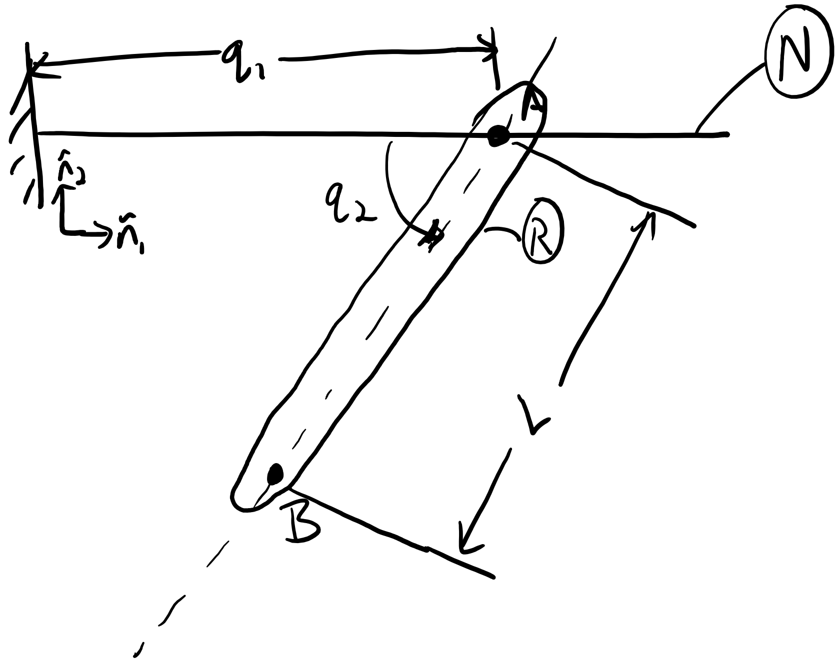
$\overline{\omega}_r, \overline{v}_r, \overline{\omega}_t$, and $\overline{v}_t$ are all functions of $q_1, \dots, q_n, t$.

Note that $\overline{\omega}_r = \dfrac{\partial \overline{\omega}}{\partial u_r}$ and $\overline{v}_r = \dfrac{\partial \overline{v}}{\partial u_r}$

$\overline{\omega}_r$ and $\overline{v}_r$ are the sensitivities of $\overline{\omega}$ and $\overline{v}$ to u_r.

Example

2 GC's



$n = 2$ 2 GC's

$m = 0$

$p = 2$ 2 DoF

$^N\bar{V}^A = \dot{q}_1 \hat{n}_1$

$^N\bar{V}^B = \dot{q}_1 \hat{n}_1 + \dot{q}_2 \hat{n}_3 \times (-Lc_2 \hat{n}_1 - Ls_2 \hat{n}_2)$

$^N\bar{V}^B = (\dot{q}_1 + L\dot{q}_2 s_2)\hat{n}_1 - L\dot{q}_2 c_2 \hat{n}_2$

$^N\bar{\omega}^R = \dot{q}_2 \hat{n}_3$

Choose some GS's.

$u_1 = \dot{q}_1$, $u_2 = \dot{q}_2$

$^N\bar{\omega}^R = u_2 \hat{n}_3$ $\quad ^B\bar{V}^A = u_1 \hat{n}_1$

$^N V^B = (u_1 + Lu_2 s_2)\hat{n}_1 - Lu_2 c_2 \hat{n}_2 = u_1 \hat{n}_1 + u_2 (Ls_2 \hat{n}_1 - Lc_2 \hat{n}_2)$

Calculate the partial velocities

	u_1	u_2
$^N\bar{\omega}^R$	0	$\hat{n}_3$
$^N\bar{V}^A$	$\hat{n}_1$	0

${}^N\bar{V}^A$	$\hat{n}_1$	0
${}^N\bar{V}^B$	$\hat{n}_1$	$L s_2 \hat{n}_1 - L c_2 \hat{n}_2$

${}^N\bar{V}_2^B \triangleq$ 2nd partial velocity of B in N